Frequency design

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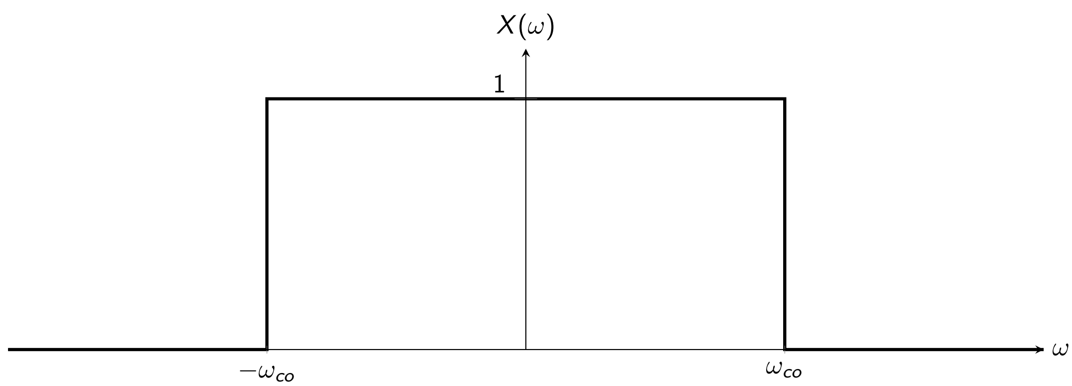
We have seen in the previous lecture that if we input a signal $x(t)$ of spectrum $X(\omega)$ into an LTI system of frequency response $H(\omega)$, then the spectrum of the corresponding output is given by the simple identity $Y(\omega) = H(\omega)X(\omega)$. Therefore, we can use this frequency point of view to design the frequency response according to what we want to keep, discard or modify in the input spectrum. We apply this idea to ideal filters and amplitude modulation.

1 Ideal filters

Ideal filters are simply systems that do not change the input spectrum, i.e. $H(\omega) = 1$, in some ranges of frequencies, and discard all the others, i.e. $H(\omega) = 0$.

Definition 1.1 (Ideal lowpass filter)
An **ideal lowpass filter** with **cutoff frequency** $f_{co} > 0$, i.e. with **cutoff impulse** $\omega_{co} = 2\pi f_{co} > 0$, is the LTI system whose frequency response is $H(\omega) = R_{-\omega_{co},\omega_{co}}(\omega) = \chi_{[-\omega_{co},\omega_{co}]}(\omega)$.



Proposition 1.1
The impulse response of an ideal lowpass filter with cutoff impulse ω_{co} is

$$\forall t \in \mathbb{R} \qquad h(t) = \frac{\omega_{co}}{\pi}\,\text{sinc}(\omega_{co} t)$$

Proof : Since the impulse response is the inverse Fourier transform of the frequency response, we have for any $t \in \mathbb{R}$,

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_{co}}^{\omega_{co}} e^{i\omega t}d\omega = \frac{1}{2\pi}\left[\frac{e^{i\omega t}}{it}\right]_{-\omega_{co}}^{\omega_{co}} = \frac{1}{2\pi}\frac{e^{i\omega_{co} t} - e^{-i\omega_{co} t}}{it} \\ &= \frac{\sin(\omega_{co} t)}{\pi t} = \frac{\omega_{co}}{\pi}\,\text{sinc}(\omega_{co} t) \end{aligned}$$

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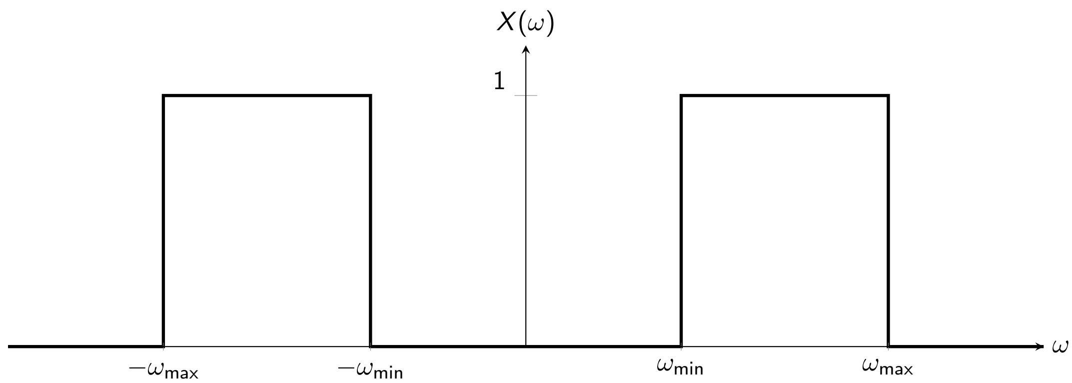
Remark: This lowpass filter is ideal because it perfectly preserves frequencies smaller that the cutoff frequency, while it discards the larger ones. However, this proposition shows that $h(t) \neq 0$ for $t < 0$, making this filter non-causal, thus not implementable in practice.
The RC circuit that we have studied in the previous lectures acts as a non-ideal lowpass filter. Indeed, we recall that its frequency response is $H(\omega) = \dfrac{1}{1 + i\omega\tau}$. Therefore, $\lim_{\omega \to 0} |H(\omega)| = 1$ and $\lim_{|\omega| \to +\infty} |H(\omega)| = 0$, which is the behavior expected for a lowpass filter.
We will see in future lectures other implementable causal lowpass filters.

Definition 1.2 (Ideal bandpass filter)
An **ideal bandpass filter** with **minimal cutoff impulse** $\omega_{\text{min}} \geq 0$ and **maximal cutoff impulse** $\omega_{\text{max}} > \omega_{\text{min}}$ is the LTI system whose frequency response is

$$H(\omega) = R_{-\omega_{\text{max}}, -\omega_{\text{min}}}(\omega) + R_{\omega_{\text{min}}, \omega_{\text{max}}}(\omega) = \chi_{[-\omega_{\text{max}}, -\omega_{\text{min}}] \cup [\omega_{\text{min}}, \omega_{\text{max}}]}(\omega)$$



Proposition 1.2
The impulse response of the ideal bandpass filter with minimal cutoff impulse ω_{min} and maximal cutoff impulse ω_{max} is

$$\forall t \in \mathbb{R} \qquad h(t) = \frac{\omega_{\text{max}}}{\pi} \operatorname{sinc}(\omega_{\text{max}} t) - \frac{\omega_{\text{min}}}{\pi} \operatorname{sinc}(\omega_{\text{min}} t)$$

Proof : The frequency response of this ideal bandpass filter can also be written:

$$H(\omega) = R_{-\omega_{\text{max}}, \omega_{\text{max}}}(\omega) - R_{-\omega_{\text{min}}, \omega_{\text{min}}}(\omega)$$

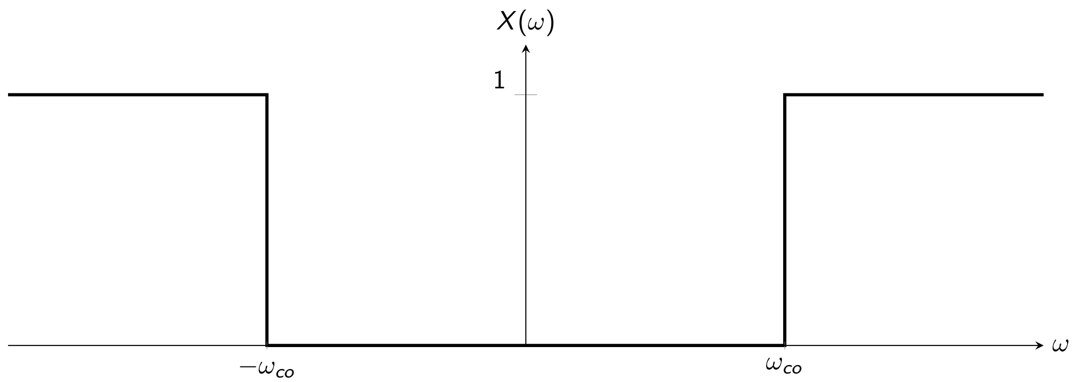
By linearity of the inverse Fourier transform, for any $t \in \mathbb{R}$,

$$h(t) = \mathcal{F}^{-1}(R_{-\omega_{\text{max}}, \omega_{\text{max}}})(t) - \mathcal{F}^{-1}(R_{-\omega_{\text{min}}, \omega_{\text{min}}})(t) = \frac{\omega_{\text{max}}}{\pi} \operatorname{sinc}(\omega_{\text{max}} t) - \frac{\omega_{\text{min}}}{\pi} \operatorname{sinc}(\omega_{\text{min}} t) \qquad \blacksquare$$

Remarks:

- As for the ideal lowpass filter, $h(t) \neq 0$ for some $t < 0$, making the ideal bandpass filter non-causal and non-implementable.
- An ideal lowpass filter with cutoff frequency ω_{co} is a particular ideal bandpass filter with $\omega_{\text{min}} = 0$ and $\omega_{\text{max}} = \omega_{co}$.

Definition 1.3 (Ideal highpass filter)
An **ideal highpass filter** with **cutoff impulse** $\omega_{co} = 2\pi f_{co} > 0$, is the LTI system whose frequency response is $H(\omega) = 1 - R_{-\omega_{co},\omega_{co}}(\omega) = 1 - \chi_{[-\omega_{co},\omega_{co}]}(\omega)$.



Proposition 1.3
The impulse response of the ideal highpass filter with cutoff impulse ω_{co} is

$$\forall t \in \mathbb{R} \qquad h(t) = \delta(t) - \frac{\omega_{co}}{\pi}\,\mathrm{sinc}(\omega_{co} t)$$

Proof : By linearity of the inverse Fourier transform,

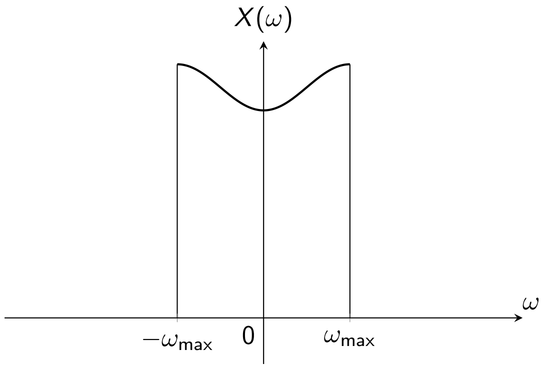
$$\forall t \in \mathbb{R} \qquad h(t) = \mathcal{F}^{-1}(\omega \mapsto 1)(t) - \mathcal{F}^{-1}(R_{-\omega_{co},\omega_{co}})(t) = \delta(t) - \frac{\omega_{co}}{\pi}\,\mathrm{sinc}(\omega_{co} t)$$ ∎

Remarks:

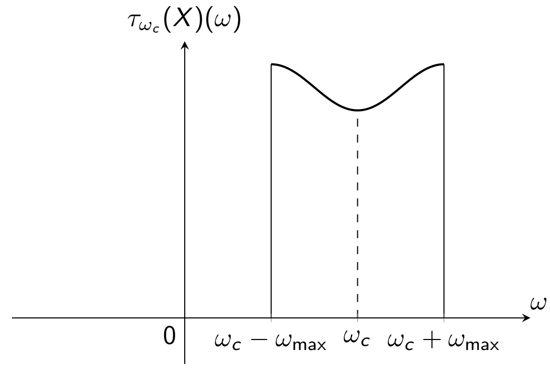
- As the ideal lowpass and bandpass filters, the ideal highpass filter is non-causal, thus non-implementable.
- An ideal highpass filter with cutoff frequency ω_{co} is a particular ideal bandpass filter with $\omega_{\min} = \omega_{co}$ and $\omega_{\max} = +\infty$.

2 Amplitude modulation

Imagine that we record a speech or sound signal which is audible by the human ear. Its frequency range is typically between 16 Hz to 16 kHz. We want to transmit this signal with over radio waves whose frenquecy range is 30 MHz to 300 MHz. We need to shift the signal spectrum to transmit it. This can be done with **amplitude modulation**.
In the following, we denote $x \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ the signal to transmit, and $X = \mathcal{F}(x)$ its Fourier transform. We assume that this spectrum is bounded, i.e. there exists $\omega_{\max} > 0$ such that X is zero outside the interval $[-\omega_{\max}, \omega_{\max}]$. With modulation, we shift this spectrum centered around 0 Hz to obtain a spectrum centered around a transmission frequency, called the **carrier frequency** and denoted f_c, corresponding to a **carrier impulse** $\omega_c = 2\pi f_c$.

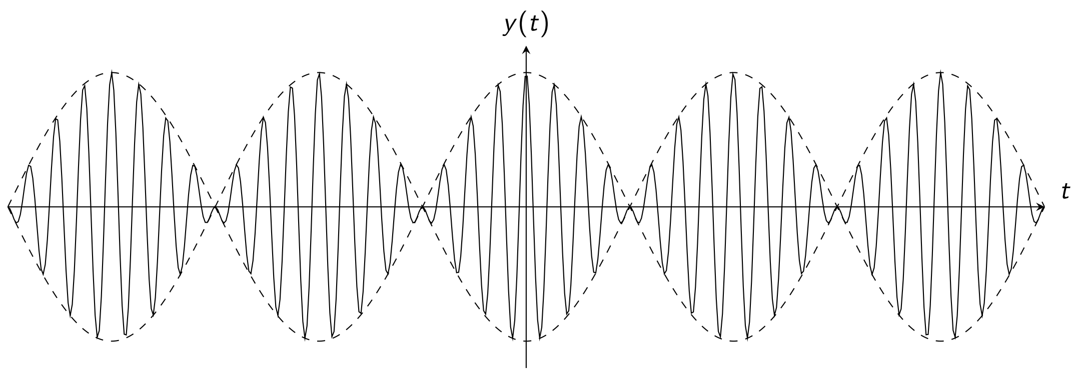


Baseband spectrum



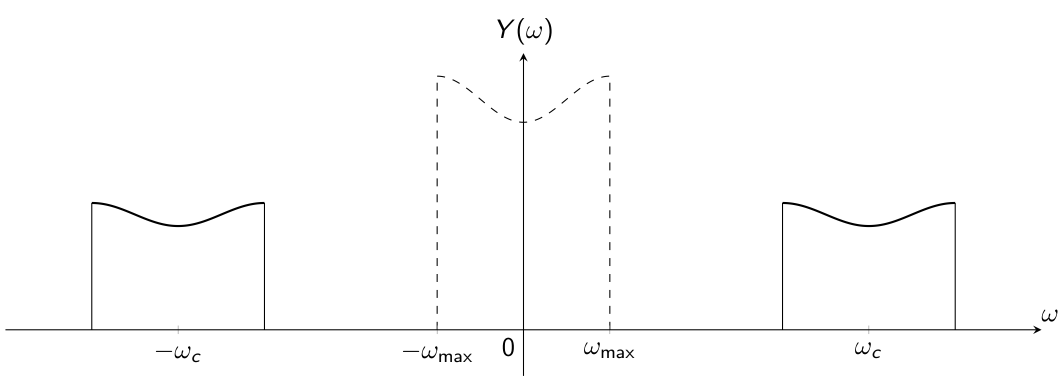
Shifted spectrum

To perform this shift, we have to convolve spectrum X by the shifted Dirac delta function δ_{ω_c}, to obtain the spectrum $Y = \delta_{\omega_c} * X = \tau_{\omega_c}(X)$. We have seen in the previous lecture that shifting in the frequency domain by ω_c corresponds to the multiplication by $e^{i\omega_c t}$ in the time domain so that the modulated signal is $y(t) = e^{i\omega_c t} x(t)$.
However, multiplying by a complex exponential implies that y is a complex-valued signal. To have a real-valued modulated signal, we rather multiply signal x by the cosine $\cos(\omega_c t) = \mathrm{Re}(e^{i\omega_c t})$, so that $y(t) = x(t)\cos(\omega_c t)$.



In the frequency domain, this gives:

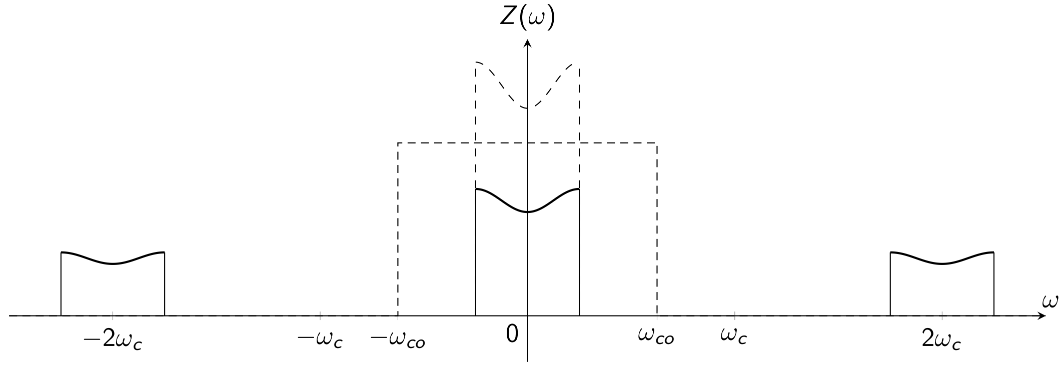
$$Y(\omega) = \mathcal{F}(xc_{\omega_c})(\omega) = \frac{1}{2\pi}\left[X * \mathcal{F}(c_{\omega_c})\right](\omega) = \frac{1}{2\pi}\left[X * (\pi\delta_{\omega_c} + \pi\delta_{-\omega_c})\right](\omega) = \frac{1}{2}X(\omega - \omega_c) + \frac{1}{2}X(\omega + \omega_c)$$



We transmit signal y through radio waves and we assume that we receive it uncorrupted by noise. As this signal was obtained by multiplying x by cosine c_{ω_c}, the intuition tells us to divide y by the same cosine to recover x. However, this is a bad idea since cosine can be zero or close to zero, yielding computation errors. Modulating x by a cosine splits its spectrum into two identical parts by shifting one half around $-\omega_c$ and the other half around ω_c. Multiplying signal y by c_{ω_c}, we find copies of X

centered around $-2\omega_c$, 0 et $2\omega_c$. Indeed, setting $z(t) = y(t)\cos(\omega_c t)$, we have:

$$Z(\omega) = \frac{1}{2}Y(\omega - \omega_c) + \frac{1}{2}Y(\omega + \omega_c) = \frac{1}{4}X(\omega - 2\omega_c) + \frac{1}{2}X(\omega) + \frac{1}{4}X(\omega + 2\omega_c)$$



Finally, to recover signal x, we have to eliminate the spectra centered around $-2\omega_c$ et $2\omega_c$ and multiply by 2 the one centered around 0. Thus we apply an ideal lowpass filter whose cutoff frequency ω_{co} is between ω_{max} and $2\omega_c - \omega_{\text{max}}$, followed by an amplifier of factor 2. Finally, we can represent the amplitude modulator and demodulator: